On the Expansion and Diameter of Bluetooth-Like Topologies

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Outline

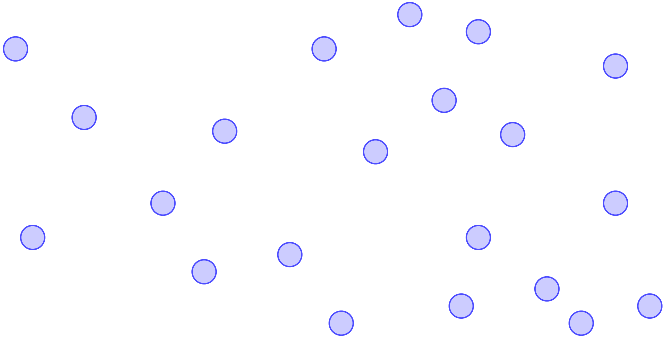
1. Technological Motivation
2. Bluetooth Topology: the Model
3. Expansion of BT
4. Diameter of BT

Bluetooth Technology

- Technology for wireless communication, named after *King Harald Bluetooth* who unified Denmark and Norway (10th century)
- Introduced as cable replacement for small PANs connecting laptops, mobile phones, PDAs, etc.
- Arguments in favor of BT for large ad-hoc scenarios:
  - cheap and easily integrable
  - good data rate/energy consumption tradeoff
  - wide adoption
  - see [Whitaker et al., 05] and [Kettimuthu, Muthukrishnan, 05]

Bluetooth Technology: Network Organization/Formation

- Transmission range r
- Piconet: 1 master, $\leqslant 7$ slaves
- Scatternet: interconnection of piconets through gateways to form multi-hop ad hoc network; three phases: device discovery, piconet formation, scatternet formation



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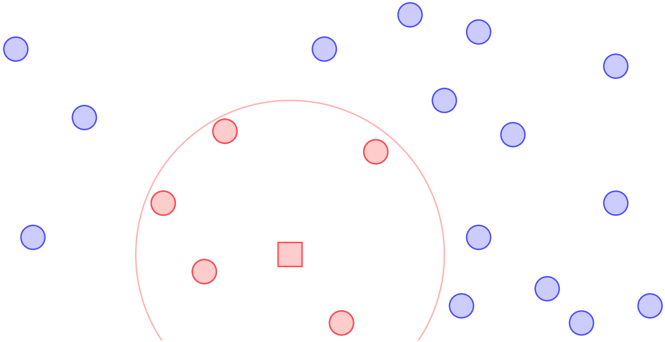

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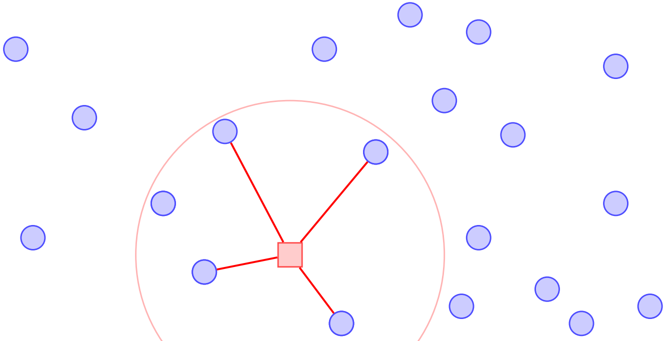

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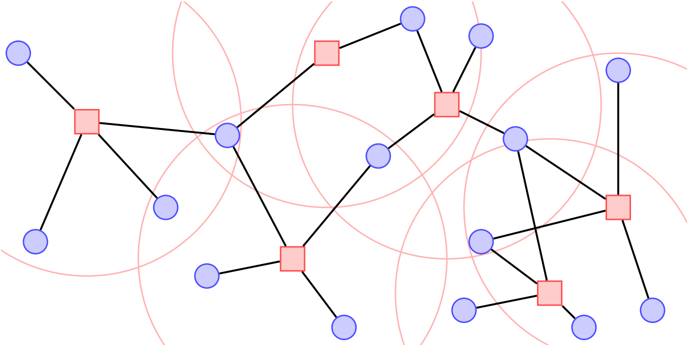

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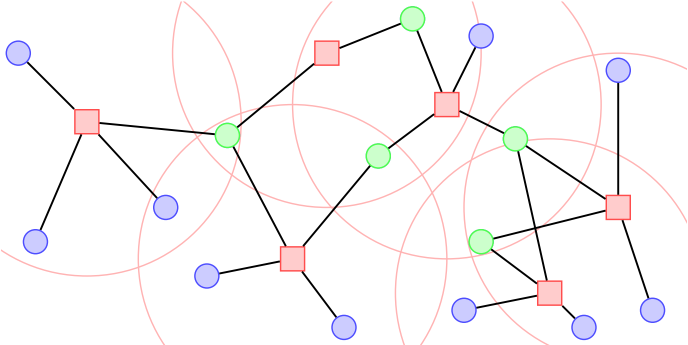


Bluetooth Technology: Device Discovery

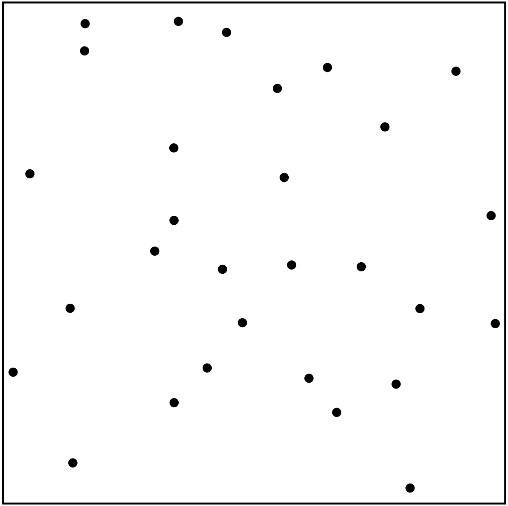
Goal

Each device must discover and set up links with (a subset of) all *visible* devices (i.e., distance $\leqslant r$) so to form a connected topology called Bluetooth Topology (BT).

Remarks:

- time and energy consuming task
- in practice, suitable time-outs (e.g., $10\,\mathrm{s}$) are used
- alternative: a node stops when at least c links have been established [Dubhashi et al., 07]

Bluetooth Topology: Mathematical Model



Graph $BT(r(n), c(n))$

- *n* nodes (*devices*) placed at random in $[0, 1]^2$

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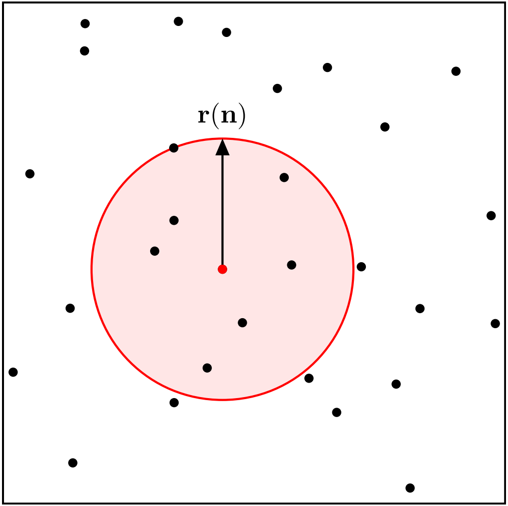

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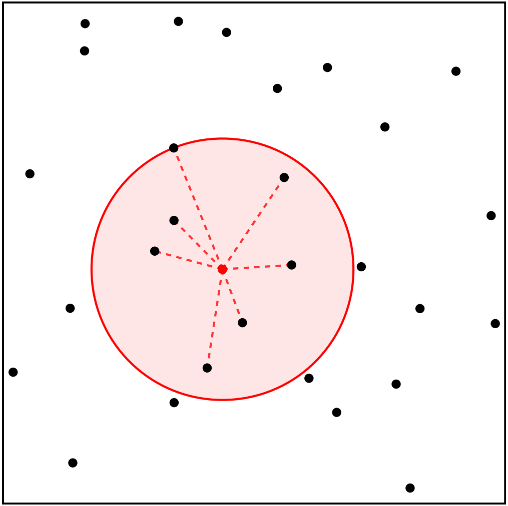

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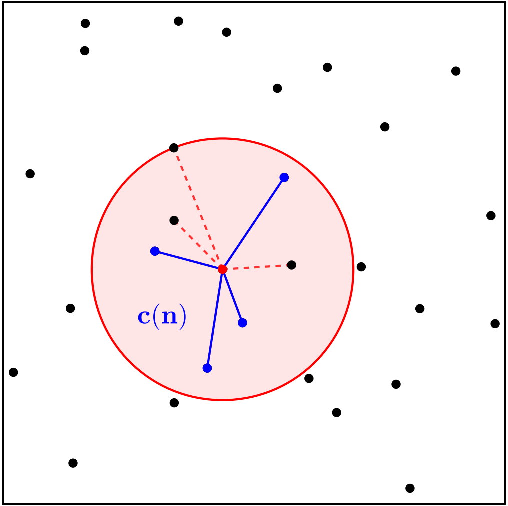
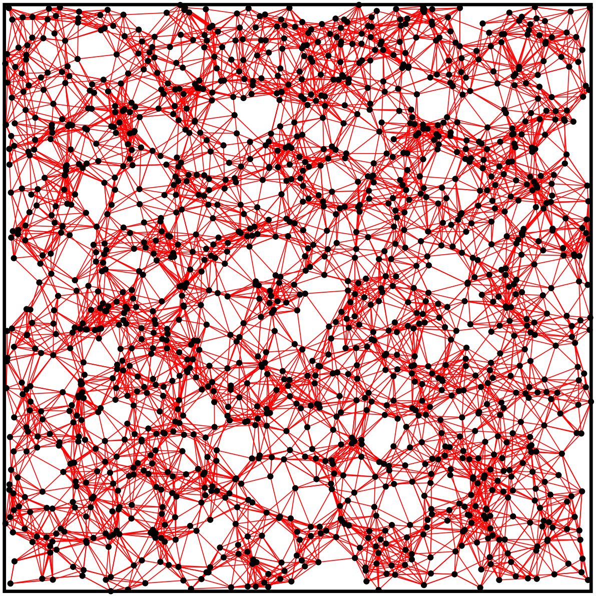


Graph $BT(r(n), c(n))$

- n nodes (*devices*) placed at random in $[0, 1]^2$
- visibility range $r(n)$
- among all visible nodes each device selects $c(n)$ random neighbors (it selects all visible nodes if $< c(n)$)



$BT(0.075, 5)$ with $n = 1500$ nodes.

Relevant Questions

How many neighbors should each device discover, in order for BT to exhibit:

- connectivity (i.e., single connected component)?
- good expansion (i.e., high bandwidth)?
- low diameter (i.e., low latency)?

Previous Work

- [Penrose 03]: $r(n) = \Omega\left(\sqrt{\ln n/n}\right)$ necessary and sufficient to achieve connectivity w.h.p., when each node connects to *all* visible nodes (Random Geometric Graph or visibility graph)
- [Panconesi et al., 04]: for $r(n) = \Theta(1)$, $c(n) = \Theta(1)$ suffices to attain high expansion w.h.p.
- [Dubhashi et al., 05]: for $r(n) = \Theta(1)$, $c(n) = 2$ suffices to attain connectivity w.h.p.

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Analysis for $r(n)$ decreasing in n is needed!

Previous Work (cont'd)

Theorem (Crescenzi, Nocentini, Pietracaprina, Pucci, 2007)

There exist two positive real constants γ_1, γ_2 such that if

$$r(n) \geqslant \gamma_1 \sqrt{\frac{\log n}{n}} \quad \textit{and} \quad c(n) = \gamma_2 \log \frac{1}{r(n)}$$

then $BT(r(n), c(n))$ is connected w.h.p.

Our Contribution

- Tight bounds on the expansion of $\mathrm{BT}(r(n), c(n))$
- Quasi-tight bounds (up to a logarithmic additive term) on the diameter of $\mathrm{BT}(r(n), c(n))$
- Results hold for all ranges of the parameters for which connectivity has been established in previous work
- We extend the results of Panconesi et al. (SPAA'04) to the case of vanishing $r(n)$

Preliminary Definitions

Let $G = (V, E)$ be an undirected connected graph.

The neighborhood of $S \subseteq V$:

$$\Gamma(S) = \{ u \in V : \exists e = (u, v) \in E,\ v \in S \}.$$

The (node) expansion of G:

$$\lambda(s) = \min_{S \subseteq V : |S| = s} \frac{|\Gamma(S) - S|}{s}, \quad 1 \leqslant s \leqslant |V|/2.$$

Expansion of BT$(r(n), c(n))$

Theorem (Expansion of BT)

Let $m = \Theta\left(nr^2(n)\right)$. Then, there exist two constants $\gamma_1, \gamma_2 > 0$ such that if

$$r(n) \geqslant \gamma_1 \sqrt{\frac{\log n}{n}} \quad \text{and} \quad c(n) = \gamma_2 \log \frac{1}{r(n)}$$

then the expansion of $BT(r(n), c(n))$ is, w.h.p.,

$$\lambda(s) = \begin{cases} \Theta\left(\min\{c(n), m/s\}\right) & \text{if} \quad 1 \leqslant s \leqslant \alpha m \\ \Theta\left(\sqrt{m/s}\right) & \text{if} \quad \alpha m < s \leqslant n/2. \end{cases}$$

Proof Roadmap

Tessellate $[0, 1]^2$ into square cells of suitable side length.

- Lower bound: for any $S \subseteq V$, $1 \leqslant |S| \leqslant |V|/2$:

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  - Characterize the local expansion of any pocket of S i.e., $S \cap$ cell;
  - Obtain expansion of S by combining pockets' expansion.
- Upper bound: select a “worst-case” subset S whose expansion matches the above lower bound (easy).

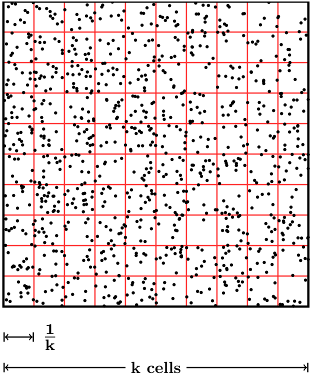
Framework

Tessellation of $[0, 1]^2$ into k^2 square cells, with

$$k = \left\lceil \frac{\sqrt{5}}{r} \right\rceil$$

(for brevity, $r \equiv r(n)$).

$\Rightarrow$ Nodes in adjacent cells are at distance $\leqslant r$.



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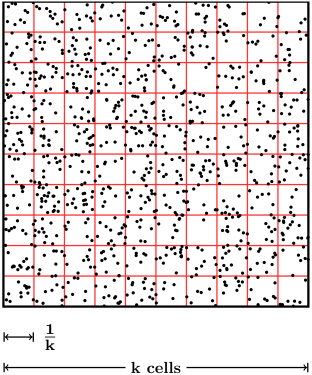
Let $\alpha = 9/10$, $\beta = 11/10$ and $m = n/k^2$.

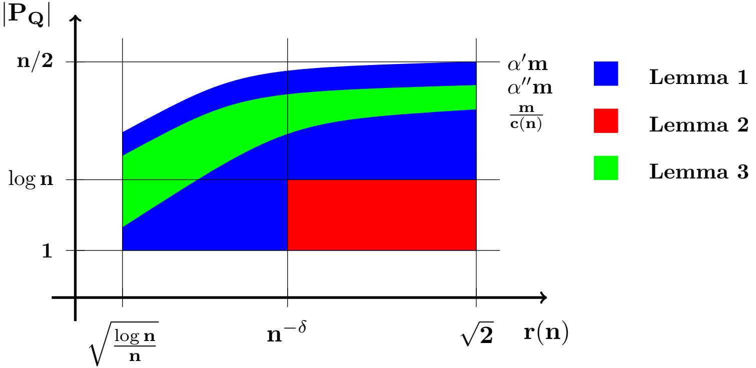
- $\Rightarrow$ W.h.p. each cell contains $\geqslant \alpha m$ and $\leqslant \beta m$ nodes.
- $\Rightarrow$ W.h.p. each device *sees* $\Theta\left(nr^2\right)$ nodes.



Lower Bounds on Pockets' Expansion

Consider a generic pocket $P_Q = S \cap Q$, for a given cell Q:

Three lemmas cover untargeted expansion for large (1) and small (2) pockets and targeted (3) expansion, respectively, as $r(n)$ varies.



The Three Lemmas

W.h.p., for every cell Q and every pocket $P \subseteq Q \cap S$, the following results hold:

Lemma 1 — Large pockets or short radii

If $|P| \geqslant \log n$ or $r(n) = O\left(n^{-1/8}\right)$, then, $\forall P : 1 \leqslant |P| \leqslant \alpha' m$,

$$|\Gamma(P) - P| \geqslant \epsilon' \min\{c(n)|P|, m\}.$$

Lemma 2 — Small pockets and large radii

If $r(n) = \Omega\left(n^{-1/8}\right)$, then, $\forall P : 1 \leqslant |P| < \log n$,

$$|\Gamma(P)| \geqslant \frac{1}{3} c(n)|P|.$$

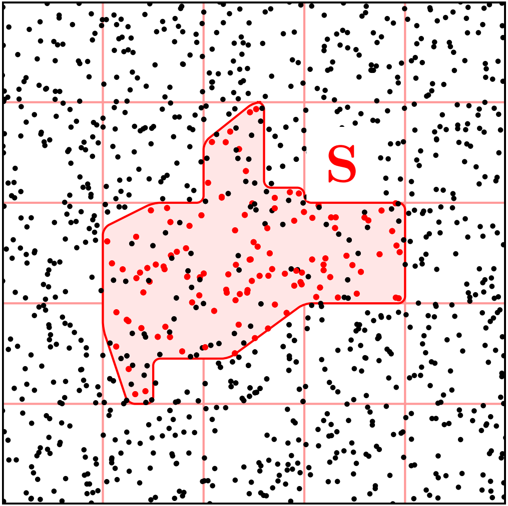
Lemma 3 — Targeted expansion

For any Q' adjacent to Q, $\forall P \subseteq Q : m/c(n) \leqslant |P| \leqslant \alpha'' m$,

$$|\Gamma(P) \cap Q'| \geqslant (1 + \epsilon'')|P|.$$

Lower Bound

Consider any $S \subseteq V$, $s = |S|$ with $1 \leqslant s \leqslant n/2$.



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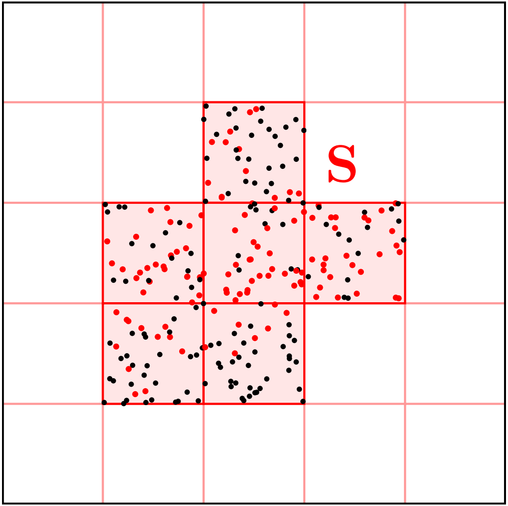

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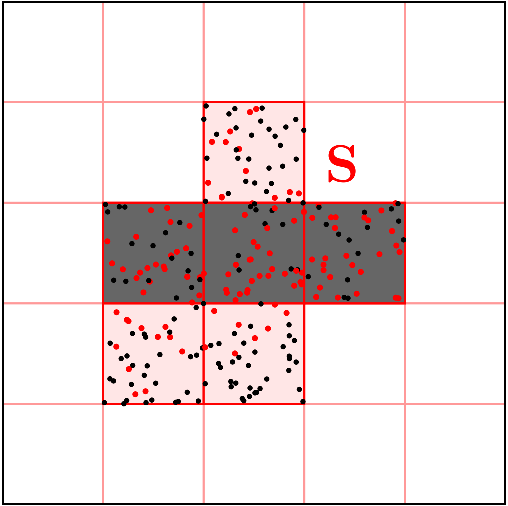

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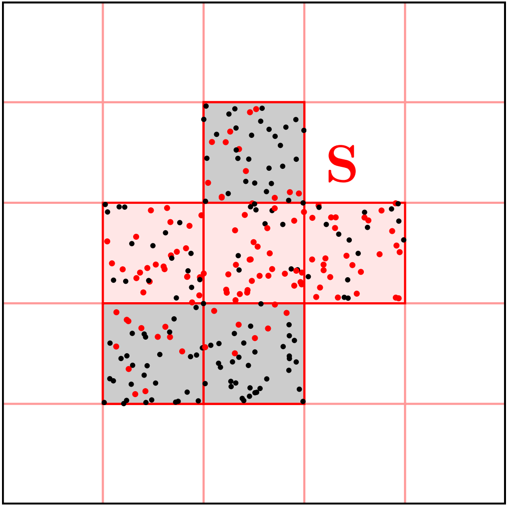

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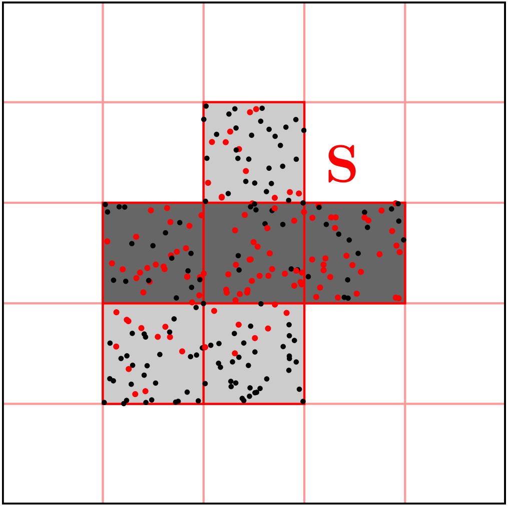
A majority of nodes of S is contained either in black or gray cells.



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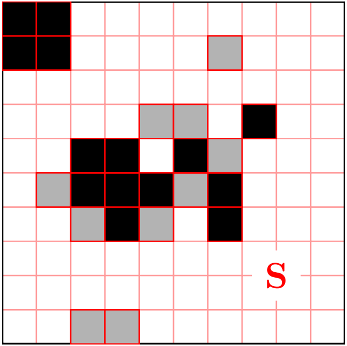

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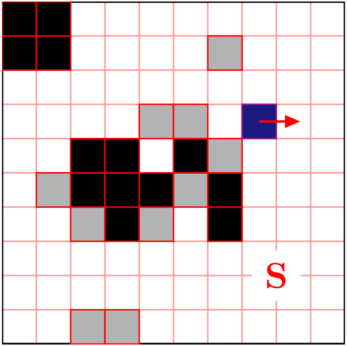

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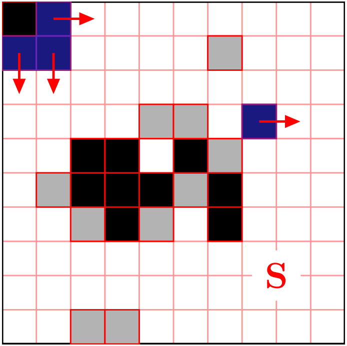

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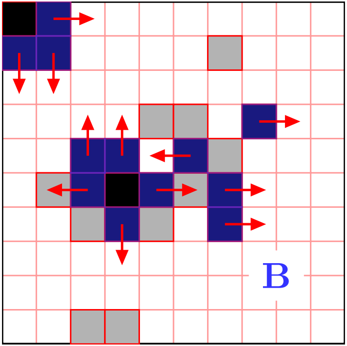

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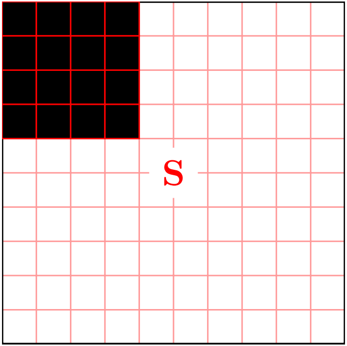

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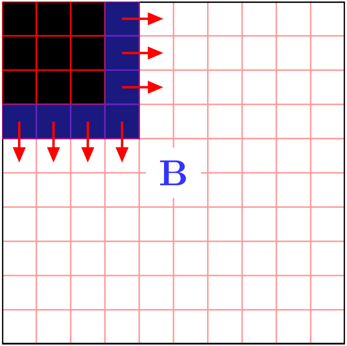

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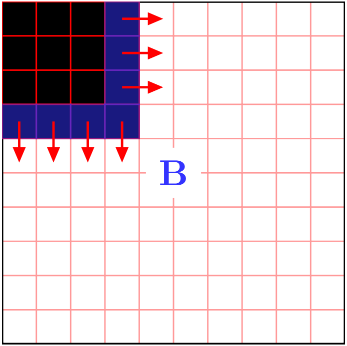
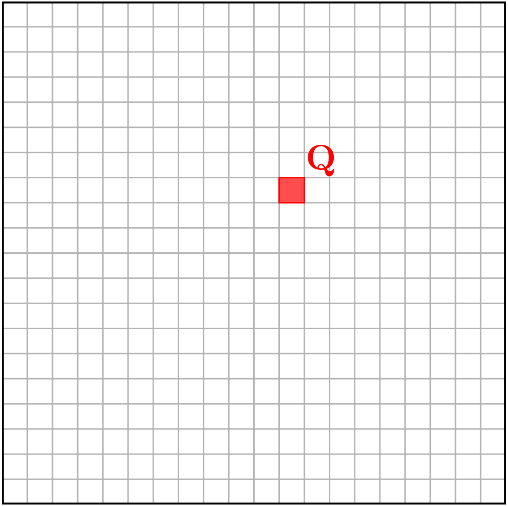


Each black/non-black pair contributes $\Omega(m)$ "new nodes" (Lemma 3), hence

$$|\Gamma(B) - S| = \Omega\left(\sqrt{sm}\right).$$

Lower Bound — Case 2: $\geqslant s/2$ nodes in gray cells

Given a cell Q, we define its sector $\mathcal{S}_Q$ and its active area $\mathcal{A}_Q$ as the squares 13×13 and 7×7 whose central cell is Q.



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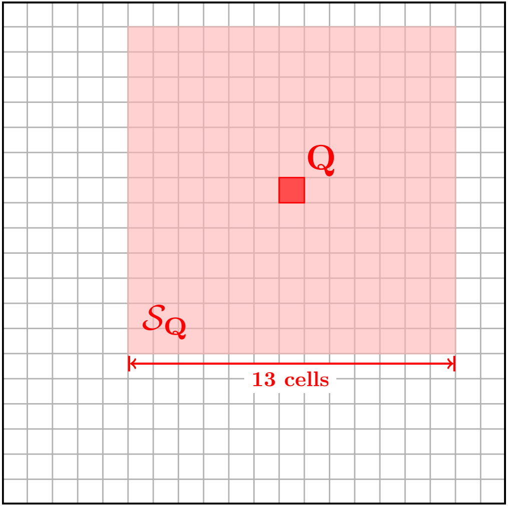

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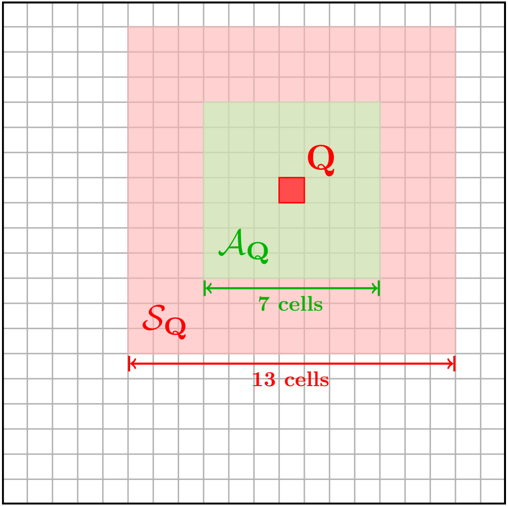

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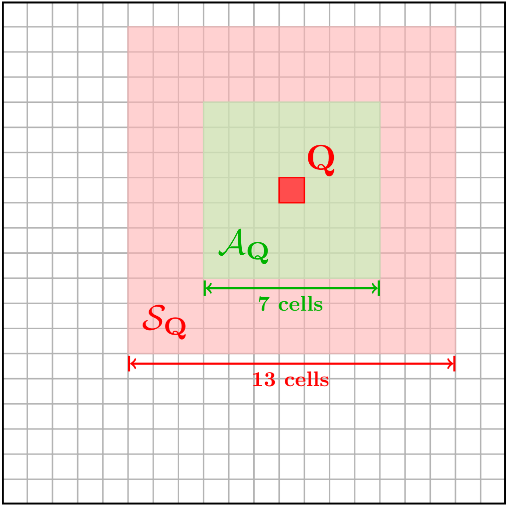

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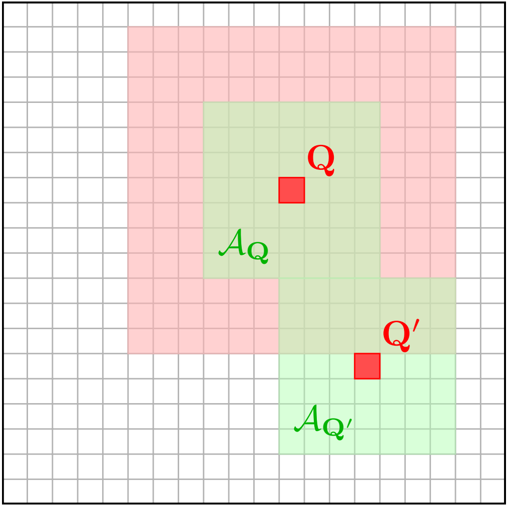
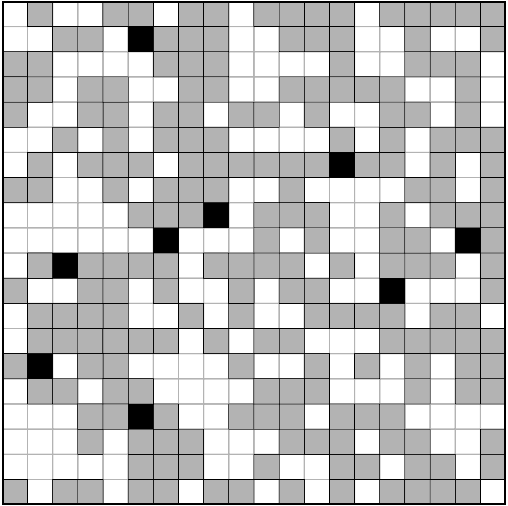
- $\forall Q' \notin \mathcal{S}_Q$, $\mathcal{A}_Q \cap \mathcal{A}_{Q'} = \emptyset$.



Lower Bound — Case 2 (cont'd)

Proof based on a greedy selection process on $S \subseteq V$.



Lower Bound — Case 2 (cont'd)

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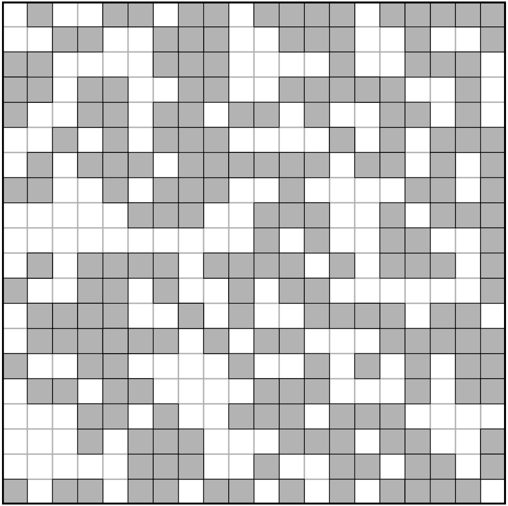
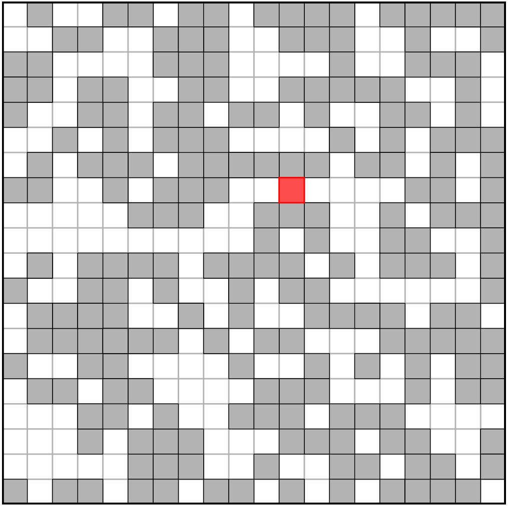


Lower Bound — Case 2 (cont'd)

Proof based on a greedy selection process on $S \subseteq V$.

1) Ignore the black cells.

2) Pick the gray cell Q with the largest pocket P



Lower Bound — Case 2 (cont'd)

Proof based on a greedy selection process on $S \subseteq V$.

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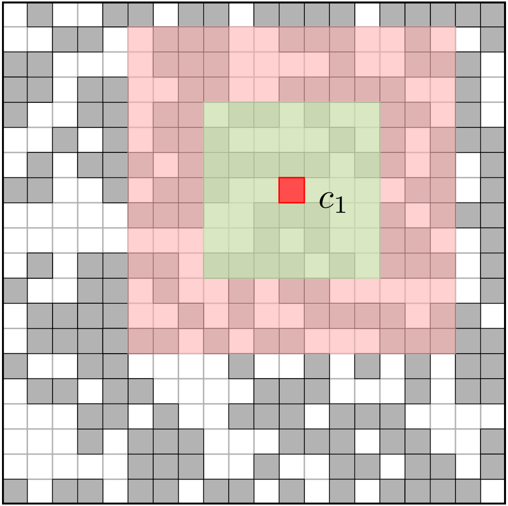

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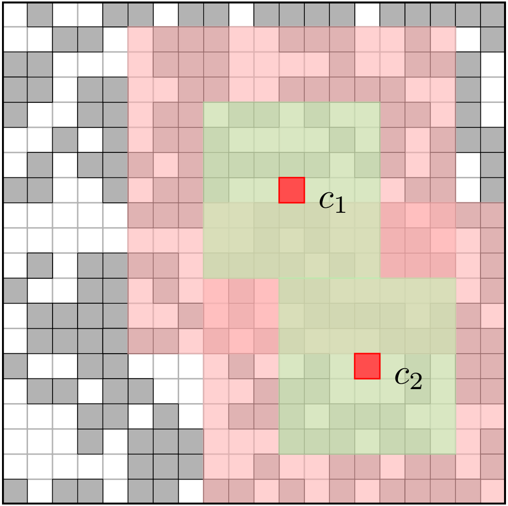

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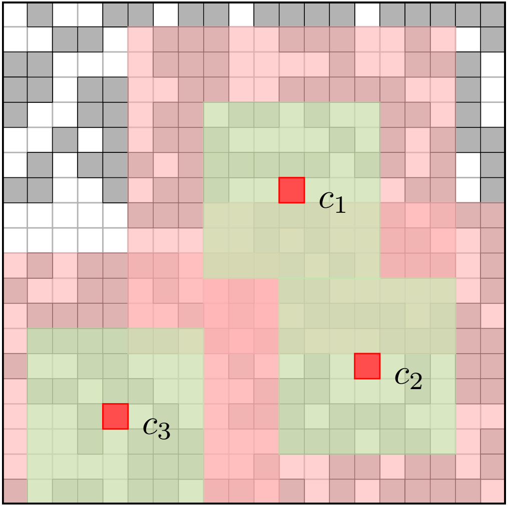

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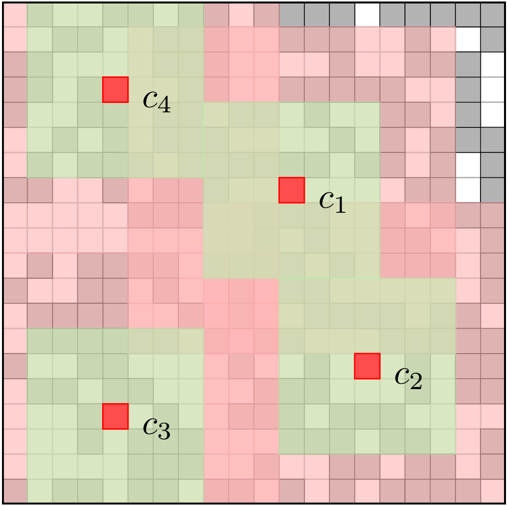

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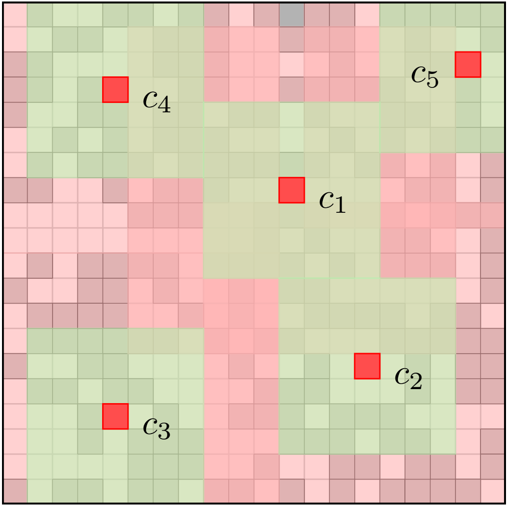

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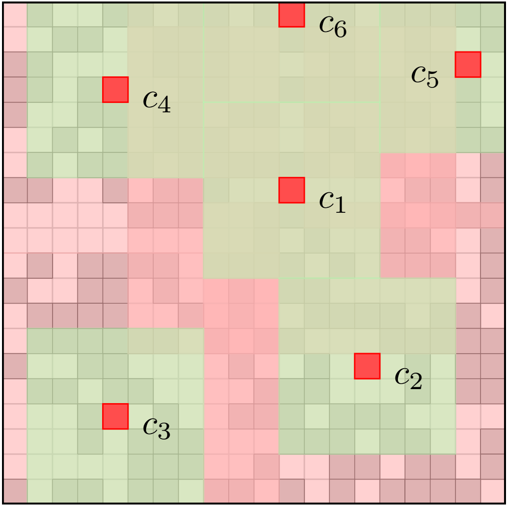

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Remark

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Details

Diameter of BT$(r(n), c(n))$

Theorem (Diameter of BT)

There exist two positive real constants γ_1, γ_2 such that if

$$r(n) \geqslant \gamma_1 \sqrt{\frac{\log n}{n}} \quad \textit{and} \quad c(n) = \gamma_2 \log \frac{1}{r(n)}$$

then the diameter of $BT(r(n), c(n))$ is, w.h.p.,

- $\mathrm{diam}(BT) = O\left(1/r(n) + \log n\right)$
- $\mathrm{diam}(BT) = \Omega\left(1/r(n)\right)$ *(tight for $r(n) = O\left(1/\log n\right)$)*
- $\mathrm{diam}(BT) = \Omega\left(\log n / \log\log n\right)$ *for $r(n) = \Theta(1)$.*

Upper Bound on the Diameter: Proof Idea

Limit the depth of any BFS tree by leveraging on the expansion result.

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Lemma (Key Recurrence)

Given a connected undirected graph $G = (V, E)$ with n nodes and expansion $\lambda(s)$, for $1 \leqslant s \leqslant n/2$, consider the following recurrence:

$$\begin{aligned} N_0 &= 1 \\ N_i &= (1 + \lambda(N_{i-1}))\, N_{i-1}. \end{aligned}$$

Define $i^\star$ as the smallest index such that $N_{i^\star} > n/2$. Then, $\mathrm{diam}(G) \leqslant 2i^\star$.

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The Theorem follows easily from the above Lemma and standard calculations.

Summary

- For $S : |S| = \Omega(m/c(n))$, the Bluetooth Topology $\mathrm{BT}(r(n), c(n))$ features the same expansion and (roughly) the same diameter as the (much denser) Random Geometric Graph $RGG(r(n))$.

Summary

- For $S : |S| = \Omega\left(m/c(n)\right)$, the Bluetooth Topology $\text{BT}(r(n), c(n))$ features the same expansion and (roughly) the same diameter as the (much denser) Random Geometric Graph $RGG(r(n))$.
- Open problem: Full characterization of the tradeoffs between $c(n)$ and connectivity/expansion/diameter.

Lower Bound — Details of Case 2

Notation:

- W is the set of selected centers, $|W| = w$;
- c_t is the center (cell) selected at the t-th iteration, $1 \leqslant t \leqslant w$;
- $P_{c_t} = S \cap c_t$;
- g_t: # of nodes of S in unmarked gray cells of $\mathcal{S}_{c_t}$ at the beginning of iteration t.

In order to lower bound the expansion of S, for all $1 \leqslant t \leqslant w$, we determine a suitably large set of nodes $N_t \subseteq \Gamma(S)$, which belong to non-black cells of $\mathcal{A}_{c_t}$.

Lower Bound — Details of Case 2 (cont'd)

Two cases are possible.

1. $\mathcal{A}_{c_t}$ contains only gray cells.

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Remark

Note that the N_t's are all disjoint, but the sum of their sizes does not immediately yield a lower bound to $|\Gamma(S) - S|$, since each N_t may itself contain nodes of S, which have to be subtracted from the overall count.

Lower Bound — Details of Case 2 (cont'd)

In the first subcase, no active area contains black cells.

The number of "new nodes" reached by G is

$$\left(\sum_{t=1}^{w} |N_t|\right) - |G| = \sum_{t=1}^{w} |N_t| - g_t \geqslant \sum_{t=1}^{w} |N_t| - 169p_t.$$

For a sufficiently large $c(n)$, we have $|N_t| - 169p_t = \mu\,|N_t|$, for a constant μ.

Hence,

$$\sum_{t=1}^{w} |N_t| - 169p_t = \Omega\left(\sum_{t=1}^{w} \bar{\epsilon}\min\{\, c(n)p_t, m\,\}\right) = \Omega\left(\min\{\, c(n)s, m\,\}\right)$$

and the theorem follows.

Lower Bound — Details of Case 2 (cont'd)

In the second subcase, some active area contains a black cell.

Partition $W = B_1 \cup B_2$ where the centers in B_1 do not have black cells in their active areas and B_2 do have.

Suppose that $\sum_{t \in B_2} |N_t| \geqslant \tau \sum_{t \in B_1} |N_t|$, where τ is a constant.

For each $t \in B_2$ the set N_t contains $(1 + \bar{\epsilon})\bar{\alpha}m$ nodes, and at least $\bar{\epsilon}\bar{\alpha}m$ of these are "new nodes". Hence, the total number of "new nodes" of S is at least

$$\sum_{t \in B_2} \bar{\epsilon}\bar{\alpha}m = \frac{\bar{\epsilon}}{1+\bar{\epsilon}} \sum_{t \in B_2} |N_t| \geqslant \frac{\bar{\epsilon}}{1+\bar{\epsilon}} \frac{\tau}{1+\tau} \sum_{t=1}^{w} |N_t| \,,$$

and the theorem follows.

Lower Bound — Details of Case 2 (cont'd)

In the second subcase, some active area contains a black cell.

Finally, if $\sum_{t \in B_2} |N_t| < \tau \sum_{t \in B_1} |N_t|$, the number of "new nodes" accounted for by the N_t's is

$$\left(\sum_{t=1}^{w} |N_t|\right) - |G| = \sum_{t \in B_1} (|N_t| - 169p_t) + \sum_{t \in B_2} (|N_t| - 169p_t)$$

$$\geqslant \sum_{t \in B_1} \mu |N_t| + \sum_{t \in B_2} ((1 + \bar{\epsilon})\bar{\alpha}m - 169\bar{\alpha}m) > \sum_{t \in B_1} \mu |N_t| - \sum_{t \in B_1} \left(\frac{169}{1 + \bar{\epsilon}} - 1\right) \tau |N_t| .$$

By fixing τ such that $((169/(1 + \bar{\epsilon})) - 1)\tau = \mu/2$, we get

$$\sum_{t \in B_1} \mu |N_t| - \sum_{t \in B_1} \left(\frac{169}{1 + \bar{\epsilon}} - 1\right) \tau |N_t| = \frac{\mu}{2} \sum_{t \in B_1} |N_t| = \Omega\left(\sum_{t=1}^{w} |N_t|\right),$$

and the theorem follows.

Return